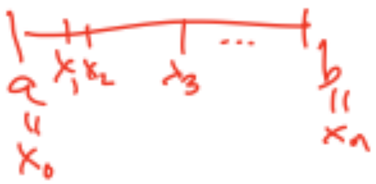
Thm (Ex 7.2.7) If $f:[a,b]$ is an increasing function, then f is Riem integrable.

(A similar proof would work for decreasing fcns.)

Rem: Can actually have an infinite # of discontinuities!

Scratch Want: $U(f,P) - L(f,P) < \varepsilon$

want $\sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \varepsilon$



$\Delta x_k = x_k - x_{k-1}$

$M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$ — $f(x_k)$

$m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}$ — $f(x_{k-1})$

$$U - L = \sum_{k=1}^{n} \big(f(x_k) - f(x_{k-1})\big)\underbrace{(x_k - x_{k-1})}_{\Delta x_k}$$

$$= \big(f(x_1) - f(x_0)\big)\Delta x_1 + \big(f(x_2) - f(x_1)\big)\Delta x_2 + \cdots + \big(f(x_n) - f(x_{n-1})\big)\Delta x_n$$

if $\Delta x_j = \frac{b-a}{n}$ $= \frac{b-a}{n}\big(\cancel{f(x_1)} - f(x_0) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cdots + f(x_n) - \cancel{f(x_{n-1})}\big)$

$$= \frac{b-a}{n}\big(-f(x_0) + f(x_n)\big) = \left(\frac{b-a}{n}\right)\big(f(b) - f(a)\big) < \varepsilon$$

want $\quad < \varepsilon$

$$\iff n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$$

use Archi.

Real Proof: With given, $\forall \varepsilon > 0$, choose $n \in \mathbb{N}$ s.t. $n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}$ (which we can do by Archie)

Then, we let P_n be the partition

$$\{a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \dots, \underbrace{a+\frac{n(b-a)}{n}}_{b}\}$$

of $[a,b]$. Then

$$U(f,P_n)-L(f,P_n) = \sum_{k=1}^{n}(M_k - m_k)\left(\frac{b-a}{n}\right),$$

where $M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\} = f(x_k)$

and $m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\} = f(x_{k-1})$.

$$\Rightarrow U(f,P_n)-L(f,P_n) = \left(\frac{b-a}{n}\right)\sum_{k=1}^{n}(f(x_k)-f(x_{k-1}))$$

$$= \left(\frac{b-a}{n}\right)(f(x_1)-f(x_0)+f(x_2)-f(x_1)+\dots+f(x_n)-f(x_{n-1}))$$

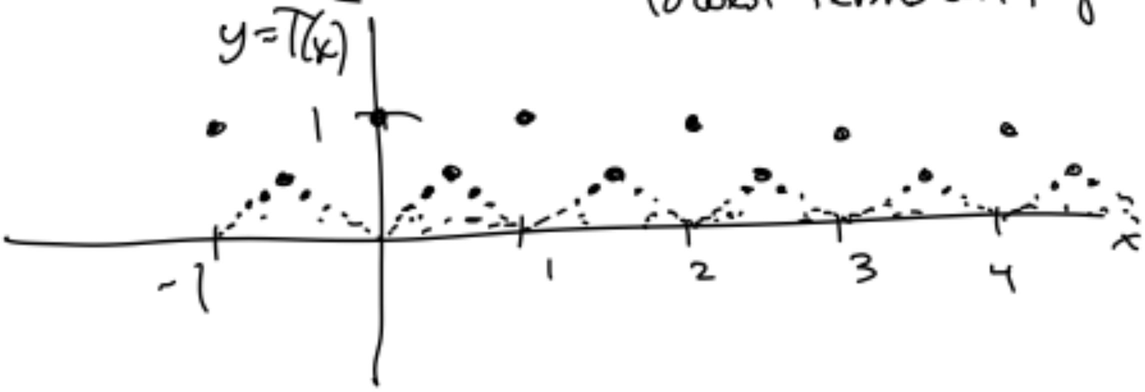
$$= \left(\frac{b-a}{n}\right)(-f(x_0)+f(x_n)) = \frac{(b-a)(f(b)-f(a))}{n}$$

$$< \frac{(b-a)(f(b)-f(a))}{\left(\frac{(b-a)(f(b)-f(a))}{\varepsilon}\right)} = \varepsilon.$$

Thus, $U(f,P_n) - L(f,P_n) < \varepsilon$, so f is integrable on $[a,b]$. $\boxed{\text{QED}}$

Example Let $T:\mathbb{R}\to\mathbb{R}$ be the Thomae function, defined by

$$T(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms with } q > 0. \end{cases}$$



<u>Proposition</u>: T is continuous at c $\iff c \in \mathbb{Q}^c$.

<u>FREAKY!</u>

<u>Proposition</u>: T is Riem integrable on any $[a,b] \subseteq \mathbb{R}$!

$$\int_a^b T(x)\,dx = 0.$$

Properties of the Integral:

1 Linearity If f, g are Riem. int on $[a,b]$ and $c, d \in \mathbb{R}$, then

$$\int_a^b (cf+dg) = c\int_a^b f + d\int_a^b g$$

(ie $cf+dg$ is integrable!)

2 If $a<b<c$, $a,b,c \in \mathbb{R}$, and f is Riem int. on $[a,c]$, then f is also Riem. inte. on $[a,b]$ and $[b,c]$, and

$$\int_a^c f = \int_a^b f + \int_b^c f .$$

2b From 2, let's define for $a<b$

$$\int_b^a f = -\int_a^b f .$$

(consistent with defns and with 2).

3 Monotonicity If f, g are Riem. int on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f \leq \int_a^b g .$$

3A Corollary: If $m \leq f(x) \leq M$ $\forall x \in [a,b]$, then $m(b-a) \leq \int_a^b f \leq M(b-a)$.

④ If f is Riem integrable on $[a,b]$, then $|f|$ is Riem integrable on $[a,b]$,

and $\int_a^b |f| \geq \left| \int_a^b f \right|$

similar $|a_1|+|a_2|+|a_3| \geq |a_1+a_2+a_3|$

Fundamental Theorem of Calculus

Suppose F is a differentiable function on $[a,b]$, and suppose that F' is Riem integrable on $[a,b]$.

Then $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Scratch. $U(F', P_n) = \sum_{k=1}^{n} M_k \Delta x_k$

where $M_k = \sup\{F'(x) : x_{k-1} \leq x \leq x_k\}$

$F(b) - F(a) = F(\underset{x_n}{b}) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2})$

$+ \cdots + \cdots + F(x_1) - F(\underset{a}{\overset{\shortparallel}{x_0}})$

$$= \left(\frac{F(b) - F(x_{n-1})}{(b - x_{n-1})}\right)\overbrace{(b - x_{n-1})}^{\Delta x_n}$$

$$+ \frac{F(x_{n-1}) - F(x_{n-2})}{(x_{n-1} - x_{n-2})}(x_{n-1} - x_{n-2}) + \cdots$$

$$+ \cdots +$$

MVT $\rightarrow$ $F'(c_n)$ $\quad x_{n-1} \le c_n \le b$

$F'(c_{n-1})$ $\quad x_{n-2} \le c_{n-1} \le x_{n-1}$

$$= F'(c_n)\Delta x_n + F'(c_{n-1})\Delta x_{n-1} + \cdots + F'(c_1)\Delta x_1$$

$$m_j \le F'(c_j) \le M_j \quad \forall j$$

$$\underbrace{\sum m_k \Delta x_k}_{L(F', P_n)} \le F(b) - F(a) \le \underbrace{\sum M_k \Delta x_k}_{U(F', P_n)}$$

$$\int_a^b F'$$

Actual Proof of FTC: with Given, we know that there exists a sequence of partitions

P_n of $[a,b]$ such that $L(F', P_n) \le U(F', P_n)$

and $\lim_{n\to\infty} L(F', P_n) = \lim_{n\to\infty} U(F', P_n) = \int_a^b F'$.

Let $P_n = \{x_0 = a, x_1, \dots, x_{k_n} = b\}$.

with $a = x_0 < x_1 < \dots < x_{k_n - 1} < x_{k_n} = b$.

Then $\forall j$, with $1 \le j \le k_n$,

$$\frac{F(x_j) - F(x_{j-1})}{x_j - x_{j-1}} = F'(c_j)$$

for some $c_j \in (x_{j-1}, x_j)$, by MVT.

Then

$$F(x_j) - F(x_{j-1}) = F'(c_j)\Delta x_j,$$

and

$$m_j \Delta x_j \le F'(c_j)\Delta x_j \le M_j \Delta x_j$$

where

$$m_j = \inf\{F'(x) : x_{j-1} \le x \le x_j\}$$
$$M_j = \sup\{F'(x) : x_{j-1} \le x \le x_j\}.$$

Thus

$$\sum_{j=1}^{k_n} m_j \Delta x_j \le \sum_{j=1}^{k_n} F'(c_j)\Delta x_j \le \sum_{j=1}^{k_n} M_j \Delta x_j$$

$$\Rightarrow \quad L(F', P_n) \le \sum_{j=1}^{k_n} F(x_j) - F(x_{j-1}) \le U(F', P_n)$$

$$\Rightarrow L(F', P_n) \le \cancel{F(x_1)} - F(x_0) + \cancel{F(x_2)} - \cancel{F(x_1)} + \cdots + F(x_n) - \cancel{F(x_{n-1})} \le U(F', P_n)$$

$$\Rightarrow L(F', P_n) \le F(\underset{b}{x_n}) - F(\underset{a}{x_0}) \le U(F', P_n).$$

$$\lim_{n\to\infty} L(F', P_n) = \int_a^b F' = \lim_{n\to\infty} U(F', P_n),$$

by the OLT

$$\int_a^b F' \le F(b) - F(a) \le \int_a^b F'$$

So $\int_a^b F' = F(b) - F(a)$. $\blacksquare$

"Second Fundamental Theorem of Calculus"
(2FTC)

If f is Riem integrable on $[a,b]$,
then if we let

$$F(x) = \int_a^x f, \text{ then}$$

(a) F is continuous on $[a,b]$.

(b) If f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0, and.

$$F'(x_0) = \frac{d}{dx}\int_a^{x_0} f = f(x_0).$$

Riemann – Lebesgue Criterion for Riemann integrability.

Thm – A function $f : [a,b] \to \mathbb{R}$ that is bounded is Riemann integrable if and only if the S of points $x \in [a,b]$ where f is not continuous at x has measure zero.

[S is called the set of discontinuities of f on $[a,b]$.)

A set $A \subseteq \mathbb{R}$ has measure zero if $\forall \varepsilon > 0$,

$\exists$ a collection $\{I_1, I_2, \dots\}$ of intervals

s.t. $S \subseteq \bigcup_{k=1}^{\infty} I_k$ and

$$\sum_{k=1}^{\infty} \ell(I_k) \leq \varepsilon$$

length.

<u>Example</u>: Any countable set has measure zero. $\hookrightarrow \{c_1, c_2, \dots\}$.

Let $I_1 = V_{\frac{\varepsilon}{2}}(c_1)$

$I_2 = V_{\frac{\varepsilon}{2^2}}(c_2)$

$\vdots$

$I_n = V_{\frac{\varepsilon}{2^n}}(c_n)$.

$$\sum_k \ell(I_k) = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$$

Cantor Set - also has measure zero.

Quiz

1. What is the first thing you will do when finals are over?
2. In the definition of Riemann integrable,
$$U(f, P_n) = \sum_{k=1}^{n} M_k \Delta x_k,$$
what is M_k?
3. Give one definition of compact set.
$K \subseteq \mathbb{R}$, K is compact iff __________________.
4. Give one definition of closed set.
$C \subseteq \mathbb{R}$: C is closed iff ________________________.